On Lévy processes conditioned to avoid zero

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Abstract

The purpose of this paper is to construct the law of a Lévy process conditioned to avoid zero, under the sole assumptions that the point zero is regular for itself and the Lévy process is not a compound Poisson process. Two constructions are proposed, the first lies on the method of h-transformation, which requires a deep study of the associated excessive function; while in the second it is obtained by conditioning the underlying Lévy process to avoid zero up to an independent exponential time whose parameter tends to 0. The former approach generalizes some of the results obtained by Yano [23] in the symmetric case, while the second is reminiscent of [8]. We give some properties of the resulting process and we describe in some detail the alpha stable case.

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1 Introduction

The main purpose of this work is to construct Lévy processes conditioned to avoid zero. This question is relevant only when 0 is non-polar. Then the event "not hitting zero" has zero probability and hence a standard analytical approach consists on finding an adequate excessive function for the process killed at the first hitting time of zero and then use Doob's h-transformation technique. A good understanding of the associated excessive function allows us to establish analytical and pathwise properties of the constructed process. This is the approach that has been used by Yano [23], under the assumption that the Lévy process is symmetric. So, our results can be seen as a generalization of the results obtained by Yano. A probabilistic approach for constructing Lévy processes conditioned to avoid zero bears on the idea that the construction can be performed by conditioning the process not to hit zero up to an independent exponential time of parameter q, and then make $q \to 0$, so that the conditioning affect the process all over the time interval $[0, \infty)$. This is a generic approach that has been used in several contexts. See for instance Chaumont and Doney [8] and the reference therein, where the case of Lévy processes conditioned to stay positive is investigated. We will prove that in our setting this procedure gives a non-degenerate limit and that both constructions coincide.

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2 Preliminaries and main results

2.1 Notation

Let $\mathcal{D}[0,\infty)$ be the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R}\cup\{\Delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega_s = \Delta\}$, where Δ is a cemetery point. The space $\mathcal{D}[0,\infty)$ is endowed with Skorohod's topology and its Borel σ-field, $\mathcal{F}$. Moreover, let $\mathbb{P}$ be a reference probability measure on $\mathcal{D}[0,\infty)$, under which the coordinate process $X = (X_t, t \geq 0)$ is a Lévy process. We will denote by $(\mathcal{F}_t, t \geq 0)$ the completed, right continuous filtration generated by X. As usual $\mathbb{P}_x$ denotes the law of $X + x$, under $\mathbb{P}$, for $x \in \mathbb{R}$. For notational convenience, we set $\mathbb{P} = \mathbb{P}_0$. We will denote by θ the shift operator and by k the killing operator, i.e., for $\omega \in \mathcal{D}[0,\infty)$, $\theta_t\omega(s) = w(s+t)$, $s \geq 0$, and

$$k_t\omega(s) = \begin{cases} w(s), & s < t, \\ \Delta, & s \geq t. \end{cases}$$

For $t \geq 0$, we use $X \circ \theta_t$, $X \circ k_t$ to denote the functions in $\mathcal{D}[0,\infty)$ given by $\theta_t\omega(\cdot)$ and $k_t\omega(\cdot)$, $\omega \in \mathcal{D}[0,\infty)$, respectively. Throughout the paper $\psi : \mathbb{R} \to \mathbb{C}$ will denote the characteristic exponent of $(X,\mathbb{P})$, which is defined by

$$\psi(\lambda) = -\frac{1}{t}\log(\mathbb{E}[e^{i\lambda X_t}]) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}})\pi(dx), \quad \lambda \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and π denotes the Lévy measure, i.e., π is a measure satisfying $\pi(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$. We denote by P_t and U_q the transition kernel at time t and the q-resolvent of the process $(X,\mathbb{P})$.

We assume throughout the paper that

H.1 The origin is regular for itself.

H.2 $(X,\mathbb{P})$ is not a compound Poisson process.

We quote the following classical result that provide an equivalent way to verify conditions **H.1** and **H.2** in terms of the characteristic exponent ψ.

Theorem 1 (See, e.g., [6] and [16]). *The conditions **H.1** and **H.2** are satisfied if and only if*

$$\int_{\mathbb{R}} Re\left(\frac{1}{q+\psi(\lambda)}\right) d\lambda < \infty, \quad q > 0$$

and

$$\textit{either} \quad \sigma > 0 \quad \textit{or} \quad \int_{|x|<1} |x|\pi(dx) = \infty.$$

It is known that under these hypotheses, for any $q > 0$, there exists a density of the resolvent kernel that we will denote by $u_q(x,y)$:

$$U_q f(x) = \int_{\mathbb{R}} u_q(x,y) f(y) dy, \quad x \in \mathbb{R},$$

for all bounded Borel functions f. The density $u_q(x,y)$ equals $u_q(y-x)$, where u_q is a continuous function. We refer to chapter II in [4] for a proof of these results. Furthermore, from the resolvent equation

$$U_q - U_r + (q-r)U_qU_r = 0, \quad q,r>0,$$

it can be deduced that the family of functions $(u_q, q>0)$ satisfies, for all $q,r>0$ with $q\neq r$,

$$\int_{\mathbb{R}} u_q(y-x)u_r(z-y)dy = \frac{1}{q-r}[u_r(z-x)-u_q(z-x)], \quad \text{for all } z,x\in\mathbb{R}. \tag{1}$$

Let T_0 be the first hitting time of zero for X:

$$T_0 = \inf\{t>0 : X_t = 0\},$$

with $\inf\{\emptyset\}=\infty$. The process killed at T_0, $X^0 = X\circ k_{T_0}$, is given by

$$X_t^0 = \begin{cases} X_t, & t<T_0, \\ \Delta, & t\geq T_0. \end{cases}$$

For every $x\in\mathbb{R}$, we will denote by $\mathbb{P}_x^0$ the law of the killed process X^0 under $\mathbb{P}_x$. We use the notation P_t^0, U_q^0 for its transition kernel and q-resolvent, respectively. From [4, Corollary 18, p. 64], it is known that,

$$\mathbb{E}_x[e^{-qT_0}] = \frac{u_q(-x)}{u_q(0)}, \quad q>0, \quad x\in\mathbb{R}. \tag{2}$$

Hence, with help of the following well known identity:

$$U_qf(x) = U_q^0 f(x) + \mathbb{E}_x[e^{-qT_0}]U_qf(0),$$

for all f bounded Borel functions and $q>0$, we obtain the resolvent density for X^0, namely,

$$u_q^0(x,y) = u_q(y-x) - \frac{u_q(-x)u_q(y)}{u_q(0)}, \quad x,y\in\mathbb{R}. \tag{3}$$

By $\widehat{\mathbb{P}}_x$ we will denote the law of the dual process $\widehat{X} := -X$ under $\mathbb{P}_{-x}$, $x\in\mathbb{R}$. We will use the notation $\widehat{\ }$ to specify the mathematical quantities related to the dual process $\widehat{X}$. For instance, $(\widehat{P}_t, t\geq 0)$, $(\widehat{U}_q, q>0)$ are the semigroup and the resolvent of the process $\widehat{X}$, respectively. It is known that the name "dual" comes from the following duality identity. Let f, g be nonnegative and measurable functions. Then, for every $t\geq 0$

$$\int_{\mathbb{R}} P_tf(x)g(x)dx = \int_{\mathbb{R}} f(x)\widehat{P}_tg(x)dx$$

and for every $q>0$

$$\int_{\mathbb{R}} U_qf(x)g(x)dx = \int_{\mathbb{R}} f(x)\widehat{U}_qg(x)dx.$$

For the semigroup and q-resolvent of the killed process we have as a consequence of *Hunt's switching identity* (see e.g. [4, p. 47, Theorem 5]):

$$\int_{\mathbb{R}} g(x)P_t^0 f(x)dx = \int_{\mathbb{R}} f(x)\widehat{P}_t^0 g(x)dx$$

and for every $q > 0$

$$\int_{\mathbb{R}} g(x)U_q^0 f(x)dx = \int_{\mathbb{R}} f(x)\widehat{U}_q^0 g(x)dx.$$

We observe that $(\widehat{X}, \widehat{\mathbb{P}})$ satisfies also the hypotheses **H.1** and **H.2**. Thus, for any $q > 0$, there exists a continuous density $\widehat{u}_q$ of the resolvent $\widehat{U}_q$. Furthermore, u_q and $\widehat{u}_q$ are related by the equation: $\widehat{u}_q(x) = u_q(-x)$, $x \in \mathbb{R}$. Thereby, for any $q > 0$, $\widehat{\mathbb{E}}_x[e^{-qT_0}]$ and the density of $\widehat{U}_q^0$ can be written in terms of u_q as follows

$$\widehat{\mathbb{E}}_x[e^{-qT_0}] = \frac{u_q(x)}{u_q(0)}, \quad q > 0, \quad x \in \mathbb{R} \tag{4}$$

and

$$\widehat{u}_q^0(x, y) = u_q(x - y) - \frac{u_q(x)u_q(-y)}{u_q(0)}, \quad x, y \in \mathbb{R}. \tag{5}$$

Since the point zero is regular for itself, there exists a continuous local time at 0 (in fact, at any point $x \in \mathbb{R}$). We denote by $L = (L_t, t \geq 0)$ the local time at zero, which is normalized by $\mathbb{E}(\int_0^\infty e^{-t} dL_t) = 1$, and by n the excursion measure away from zero for X. The measure n has its support on the set of excursions away from zero:

$$\mathcal{D}^0 = \{\epsilon \in \mathcal{D}[0, \infty) : \epsilon(t) \neq 0, 0 < t < \zeta(\epsilon), 0 < \zeta(\epsilon) \leq \infty\}.$$

A nice relation between the excursion measure n and the Laplace transform of the law of T_0 under $\widehat{\mathbb{P}}_x$ can be found in [24, Theorem 3.3] for Lévy processes and in [14, eq. (3.22)], [10, eq. (2.8)] for general Markov processes. This is stated as follows, let f be a nonnegative measurable function, then

$$\int_0^\infty e^{-qt} n(f(X_t), t < \zeta)dt = \int_{\mathbb{R}} f(x)\widehat{\mathbb{E}}_x[e^{-qT_0}]dx. \tag{6}$$

In particular, if $f \equiv 1$,

$$\int_0^\infty e^{-qt} n(\zeta > t)dt = \frac{1}{q u_q(0)}, \quad q > 0. \tag{7}$$

2.2 Main results

Under the assumptions **H.1**, **H.2** and

H.3 $(X, \mathbb{P})$ is symmetric,

Yano [23] showed that the function h defined by

$$h(x) = \lim_{q \to 0} [u_q(0) - u_q(x)], \quad x \in \mathbb{R} \tag{8}$$

is a well defined invariant function for the semigroup of the Lévy process killed at its first hitting time of zero. Furthermore, Yano proved that the function h can be expressed in terms of the characteristic exponent of X as

$$h(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1-\cos\lambda x}{\theta(\lambda)}d\lambda,\quad x\in\mathbb{R},\tag{9}$$

where $\theta(\lambda)=Re\psi(\lambda)$. Our first main result generalizes (8) and (9).

Throughout the rest of this paper we assume that H.1 and H.2 are satisfied.

Theorem 2. *For $q>0$, let h_q denote the function defined by*

$$h_q(x)=u_q(0)-u_q(-x),\quad q>0,\quad x\in\mathbb{R}.\tag{10}$$

Then, the function $h:\mathbb{R}\to\mathbb{R}$ defined by

$$h(x)=\lim_{q\to 0}h_q(x),\quad x\in\mathbb{R}\tag{11}$$

is such that

(i) for every $x\in\mathbb{R}$, $0\le h(x)<\infty$ and

$$h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left(\frac{1-e^{i\lambda x}}{\psi(\lambda)}\right)d\lambda,\quad x\in\mathbb{R},\tag{12}$$

(ii) h is subadditive, continuous function, which vanishes only at the point $x=0$,

(iii) h is invariant with respect to the semigroup of the Lévy process killed at T_0, i.e.,

$$P_t^0h(x)=h(x),\quad t>0,\quad x\in\mathbb{R};\tag{13}$$

furthermore

$$n(h(X_t),t<\zeta)=1,\quad\forall\, t>0.$$

The proof of (i) and (ii) in Theorem 2 will be given in section 3.2, as a consequence of analogous results for the sequence of functions $(h_q)_{q>0}$. In order to establish (iii) and other results, and due to technical issues, we will introduce an auxiliary function h^*. The function h^* dominates h and satisfies some integrability conditions. This function, as its name indicates, will help us to prove the main results acting as a dominating function in the dominated convergence theorem. The function h^* is closely related to the local time of the Lévy process $(X,\mathbb{P})$, namely, we have the expression

$$h^*(x)=\mathbb{E}(L_{T_x})=\lim_{q\to 0}\mathbb{E}\left(\int_0^{T_x}e^{-qt}dL_t\right),\quad x\in\mathbb{R}.$$

The function h^* arises as a particular case of a general function $h(\cdot,\cdot)$ defined by

$$h(x,y)=\mathbb{E}_x(L^x_{T_y})=\mathbb{E}_0(L^0_{T_{y-x}})=h(0,y-x)=h^*(y-x),$$

where L_t^x denotes the local time at the point x for the process $(X, \mathbb{P}_x)$. The function $h(\cdot, \cdot)$ is used to establish continuity criteria for local times of Lévy processes, see [1, 2] for this case and [13] for a general Borel right Markov processes.

Besides, in the present context, both Yano's and our results extend the theory of invariant functions for killed Lévy processes that can be found in Section 23 of the treatise by Port and Stone [18] on the potential theory for Lévy processes in locally compact, non-compact, second countable Abelian groups. The relations with this work will be described in Section 3.3 below.

Having constructed the invariant function h, in the following definition, we introduce the associated h-process. We will show that the resulting probability measures are such that the canonical process X never hits the point zero, and thus that we refer to them as the law of the Lévy process conditioned to avoid zero. Theorem 5 below summarises these properties.

Definition 3. *We denote by* $(\mathbb{P}_x^\updownarrow, x \in \mathbb{R})$ *the unique family of measures such that for* $x \in \mathbb{R}$*,*

$$\mathbb{P}_x^\updownarrow(\Lambda) = \begin{cases} \dfrac{1}{h(x)} \mathbb{E}_x^0(\mathbf{1}_\Lambda h(X_t)), & x \neq 0, \\ \\ n(\mathbf{1}_\Lambda h(X_t) \mathbf{1}_{\{t<\zeta\}}), & x = 0, \end{cases}$$

for all $\Lambda \in \mathcal{F}_t$*, for all* $t \geq 0$*. We will refer to it as the law of* X *conditioned to avoid* 0*.*

Remark 4. Note that from this definition, $\mathbb{P}_x^\updownarrow(T_0 > t) = 1$, for all $t > 0$, $x \in \mathbb{R}$. Hence, $\mathbb{P}_x^\updownarrow(T_0 = \infty) = 1$, for all $x \in \mathbb{R}$.

Theorem 5. *The family of measures* $(\mathbb{P}_x^\updownarrow)_{x \in \mathbb{R}}$ *is Markovian and satisfies*

(i) $\mathbb{P}_x^\updownarrow(X_0 = x) = 1$, $\forall\, x \in \mathbb{R}$.

(ii) $\mathbb{P}_x^\updownarrow(T_0 = \infty) = 1$, $\forall\, x \in \mathbb{R}$.

The semigroup associated to $(\mathbb{P}_x^\updownarrow)_{x \in \mathbb{R}}$ *is given by*

$$P_t^\updownarrow(x, dy) := \frac{h(y)}{h(x)} P_t^0(x, dy), \quad x \neq 0, \quad t \geq 0.$$

The entrance law under $\mathbb{P}_0^\updownarrow$ *is given by*

$$\mathbb{P}_0^\updownarrow(X_t \in dy) = n(h(y) \mathbf{1}_{\{X_t \in dy\}} \mathbf{1}_{\{t<\zeta\}}).$$

We propose an alternative construction of the law of the Lévy process conditioned to avoid zero. Our construction is inspired from [3, 7, 8, 9], where Lévy processes conditioned to stay positive are constructed. Lévy processes conditioned to stay positive are constructed in the following way. Let $\underline{L}_t$ be the local time of the process X reflected at its past infimum, that is, $X - \underline{X}$, where $\underline{X}_t := \inf\{X_s : 0 \leq s \leq t\}$. Let $\underline{n}$ be the measure of its excursions away from zero and let $\tau_{(-\infty,0)}$ be the first hitting time of the negative half-line. Denote by

$(Q_t(x, dy), t \geq 0, x \geq 0, y \geq 0)$ the semigroup of the process killed at $\tau_{(-\infty,0)}$. In [8, 9] is proven that the function l defined by

$$l(x) := \mathbb{E}\left(\int_{[0,\infty)} \mathbf{1}_{\{\underline{X}_t \geq -x\}} d\underline{L}_t\right), \quad x \geq 0, \tag{14}$$

is an excessive or invariant function for the semigroup $(Q_t, t \geq 0)$. The function l is actually an invariant function whenever X does not drift towards infinity. Furthermore, they obtained l as a limit of certain sequence of functions. To be precise, if $\mathbf{e}_q$ is an exponential random time with parameter $q > 0$ and independent of $(X, \mathbb{P})$, then for $x \geq 0$,

$$l(x) = \lim_{q\to 0} \frac{\mathbb{P}_x(\tau_{(-\infty,0)} > \mathbf{e}_q)}{\eta q + \underline{n}(\zeta > \mathbf{e}_q)}, \tag{15}$$

where η is such that $\int_0^t \mathbf{1}_{\{X_s = \underline{X}_s\}} ds = \eta \underline{L}_t$ and $\underline{n}(\zeta > \mathbf{e}_q) = \int_0^\infty q e^{-qt} \underline{n}(\zeta > t) dt$. They also showed that the law of Lévy processes conditioned to stay positive can be obtained as a limit, as $q \to 0$, of the law of the process conditioned to stay positive up to an independent exponential time with parameter q (see Proposition 1 in [8]).

The following theorem states that for $x \neq 0$, $\mathbb{P}^\updownarrow_x$ is the limit, as $q \to 0$, of the law of the process X under $\mathbb{P}_x$ conditioned to avoid zero, up to an independent exponential time with parameter $q > 0$. Since an exponential random variable with parameter q converges in distribution to infinity as its parameter converges to zero, then this result confirms that, starting at $x \neq 0$, we can think of X under $\mathbb{P}^\updownarrow_x$, as the process conditioned to avoid zero on the whole positive real line.

Theorem 6. *Let $\mathbf{e}_q$ be an exponential time with parameter $q > 0$, independent of $(X, \mathbb{P})$. Then for any $x \neq 0$, and any $(\mathcal{F}_t)_{t\geq 0}$-stopping time T,*

$$\lim_{q\to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathbb{P}^\updownarrow_x(\Lambda), \quad \forall\, \Lambda \in \mathcal{F}_T.$$

In the case $x = 0$, the law $\mathbb{P}^\updownarrow_0$ can also be obtained as a limit involving an independent exponential time. Before stating the result, we point out that for $s > 0$, we will denote by $g_s = \sup\{t \leq s : X_t = 0\}$, the last zero of X before time s.

Proposition 7. *Let $\mathbf{e}_q$ be an exponential time with parameter $q > 0$, independent of $(X, \mathbb{P})$. Let $\mathbb{P}^{\mathbf{e}_q}$ be the law of $X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}} \circ \theta_{g_{\mathbf{e}_q}}$ under $\mathbb{P}$. Then, for $t > 0$,*

$$\lim_{q\to 0} \mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = \mathbb{P}^\updownarrow_0(\Lambda) = n(\mathbf{1}_\Lambda h(X_t) \mathbf{1}_{\{t<\zeta\}}), \quad \forall\, \Lambda \in \mathcal{F}_t.$$

Another important property of the h-process is its transiency. This is given in the following proposition.

Proposition 8 (Transiency property)**.** *The process $(X, \mathbb{P}^\updownarrow_x)_{x\in\mathbb{R}}$ is transient.*

In Lemma 24 we will prove that for any $x \neq 0$, the point x is regular for itself under $\mathbb{P}^\updownarrow_x$. Therefore, there exists a local time at any point $x \in \mathbb{R} \setminus \{0\}$, and we will denote by $n^\updownarrow_x$ the excursion measure away from x for the process $(X, \mathbb{P}^\updownarrow_x)$. In the following proposition we establish a relationship between the excursion measure away from zero for $(X, \mathbb{P})$ and the excursion measure away from x for $(X, \mathbb{P}^\updownarrow_x)$, $x \neq 0$.

Proposition 9. *For $x \neq 0$, let $n_x^\updownarrow$ be the excursion measure out from x for $(X, \mathbb{P}_x^\updownarrow)$ and n the excursion measure out from zero for $(X, \mathbb{P})$. Then, for any measurable and bounded functional $H : \mathcal{D}^0 \to \mathbb{R}$,*

$$n_x^\updownarrow \left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt \right) = \frac{1}{h(x)} n \left(\int_0^\zeta H(\epsilon_u + x, u < t) h(X_t + x) \mathbf{1}_{\{T_{\{-x\}} > t\}} q e^{-qt} dt \right).$$

3 Proofs

3.1 A preliminary result

In order to prove the finiteness of h, we need the following lemma.

Lemma 10. *Let $(X, \mathbb{P})$ be a Lévy process with characteristic exponent ψ. Assume that $(X, \mathbb{P})$ satisfies the hypotheses **H.1** and **H.2**, then, $\psi(\lambda) \neq 0$, for all $\lambda \neq 0$ and*

$$\lim_{|\lambda| \to \infty} \psi(\lambda) = \infty.$$

Furthermore,

$$\int_{\mathbb{R}} (1 \wedge \lambda^2) Re \left(\frac{1}{\psi(\lambda)} \right) d\lambda < \infty. \tag{16}$$

Proof. The first part follows from the fact that $(X, \mathbb{P})$ is not arithmetic (see e.g. [11, Theorem 6.4.7]). Now, since $1/(1 + \psi)$ is the Fourier transform of the integrable function u_1, then from the Riemann-Lebesgue theorem it follows $\lim_{|\lambda| \to \infty} \psi(\lambda) = \infty$.

Using that $\lim_{|\lambda| \to \infty} \psi(\lambda) = \infty$, we deduce

$$Re \left(\frac{1}{\psi(\lambda)} \right) \sim Re \left(\frac{1}{1 + \psi(\lambda)} \right), \quad |\lambda| \to \infty.$$

The latter and Theorem 1 imply that for all $\lambda_0 > 0$,

$$\int_{|\lambda| > \lambda_0} Re \left(\frac{1}{\psi(\lambda)} \right) d\lambda < \infty. \tag{17}$$

On the other hand,

$$\begin{aligned} \left[Re \left(\frac{\lambda^2}{\psi(\lambda)} \right) \right]^{-1} &\geq \frac{Re \psi(\lambda)}{\lambda^2} \\ &\geq \sigma^2 + \int_{|y| < 1} \frac{(1 - \cos \lambda y)}{\lambda^2} \pi(dy) \\ &\longrightarrow \sigma^2 + \int_{|y| < 1} y^2 \pi(dy) > 0, \quad \text{as } \lambda \to 0. \end{aligned}$$

The latter limit implies that there exists a λ_0 such that,

$$Re \left(\frac{\lambda^2}{\psi(\lambda)} \right) \leq C, \quad \text{for all } |\lambda| < \lambda_0, \tag{18}$$

for some constant positive C. Then, from (17) and (18), we obtain (16). □

3.2 Some properties of h_q and h

In order to establish some properties of h, we write h_q in an alternative form, namely in terms of T_0 and the excursion measure n, as follows. Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$ and independent of $(X, \mathbb{P})$. Using (2) and (7), we can write

$$\begin{aligned} h_q(x) &= u_q(0)(1 - \mathbb{E}_x(e^{-qT_0})) \\ &= \frac{\mathbb{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}, \end{aligned} \tag{19}$$

where

$$n(\zeta > \mathbf{e}_q) = \int_0^\infty q e^{-qt} n(\zeta > t) dt = \frac{1}{u_q(0)}.$$

The expression (19) helps us to prove the following lemma, which summarizes some important properties of the sequence $(h_q)_{q>0}$.

Lemma 11. *For every $q > 0$, the function h_q is subadditive on $\mathbb{R}$ and it is excessive for the semigroup $(P_t^0, t \geq 0)$.*

Proof. By Proposition 43.4 in [20], we have that for any $q > 0$ and $x, y \in \mathbb{R}$,

$$\mathbb{E}_{x+y}(e^{-qT_0}) \geq \mathbb{E}_x(e^{-qT_0})\mathbb{E}_y(e^{-qT_0}). \tag{20}$$

Now, since

$$(1 - \mathbb{E}_x(e^{-qT_0}))(1 - \mathbb{E}_y(e^{-qT_0})) \geq 0,$$

then using (20), it follows

$$1 - \mathbb{E}_x(e^{-qT_0}) + 1 - \mathbb{E}_y(e^{-qT_0}) \geq 1 - \mathbb{E}_{x+y}(e^{-qT_0}).$$

Hence, by (19)

$$h_q(x + y) \leq h_q(x) + h_q(y), \quad x, y \in \mathbb{R}.$$

This shows that h_q is subadditive on $\mathbb{R}$.

In order to show that h_q is excessive for P_t^0, we claim that

$$\mathbb{P}_x(T_0 > t + \mathbf{e}_q) = \mathbb{E}_x\left(\mathbf{1}_{\{T_0 > t + \mathbf{e}_q\}}\right) = \mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{t < T_0\}}\right). \tag{21}$$

Indeed, we note that for $t > 0$ fixed, $T_0 \circ \theta_t + t = T_0$, on $\{T_0 > t\}$. From this remark and the Markov property, we obtain the following identities

$$\begin{aligned} \mathbb{P}_x(T_0 > t + \mathbf{e}_q) &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > t+s\}}) q e^{-qs} ds \\ &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > t+s\}}\mathbf{1}_{\{T_0 > t\}}) q e^{-qs} ds \\ &= \int_0^\infty \mathbb{E}_x(\mathbf{1}_{\{T_0 > s\}} \circ \theta_t \mathbf{1}_{\{T_0 > t\}}) q e^{-qs} ds \\ &= \mathbb{E}_x\left(\mathbf{1}_{\{T_0 > \mathbf{e}_q\}} \circ \theta_t \mathbf{1}_{\{T_0 > t\}}\right) \\ &= \mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{T_0 > t\}}\right). \end{aligned}$$

The identities (21) and (19) imply

$$\begin{aligned}\mathbb{E}_x(h_q(X_t), t < T_0) &= \mathbb{E}_x\left(\frac{\mathbf{1}_{\{T_0 > t + \mathbf{e}_q\}}}{n(\zeta > \mathbf{e}_q)}\right) \\ &\leq \mathbb{E}_x\left(\frac{\mathbf{1}_{\{T_0 > \mathbf{e}_q\}}}{n(\zeta > \mathbf{e}_q)}\right) \\ &= h_q(x).\end{aligned}$$

The above expression also implies that $\lim_{t\to 0}\mathbb{E}_x(h_q(X_t), t < T_0) = h_q(x)$, for $x \in \mathbb{R}$. This shows that h_q is excessive for the semigroup $(P_t^0, t \geq 0)$. $\square$

Before we proceed to the proof of (i) and (ii) in Theorem 2 we make a technical remark.

Remark 12. Proceeding as in the proof of Theorem 19 p. 65 in [4], it can be shown that

$$u_q(x) = \frac{1}{2\pi}\int_{\mathbb{R}} Re\left(\frac{e^{-i\lambda x}}{q + \psi(\lambda)}\right) d\lambda, \quad x \in \mathbb{R}. \tag{22}$$

Then,

$$2u_q(0) - [u_q(x) + u_q(-x)] = \frac{1}{\pi}\int_{\mathbb{R}}(1 - \cos\lambda x) Re\left(\frac{1}{q + \psi(\lambda)}\right) d\lambda.$$

On the other hand, making use of the inequality $|1 - \cos b| \leq 2(1 \wedge b^2)$ and (16), we obtain

$$\int_{\mathbb{R}}(1 - \cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda < \infty, \quad x \in \mathbb{R}.$$

Therefore, for all $x \in \mathbb{R}$,

$$\lim_{q\to 0}(2u_q(0) - [u_q(x) + u_q(-x)]) = \frac{1}{\pi}\int_{\mathbb{R}}(1 - \cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda \tag{23}$$

is finite.

Proof of (i) and (ii) in Theorem 2. That h is subadditive and excessive follow from Lemma 11 (since these properties are preserved under limits of sequences of functions).

To obtain the finiteness of h, we note that for all $q > 0$, $x \in \mathbb{R}$,

$$h_q(x) \leq 2u_q(0) - [u_q(x) + u_q(-x)] = \frac{1}{\pi}\int_{-\infty}^{\infty}(1 - \cos\lambda x) Re\left(\frac{1}{q + \psi(\lambda)}\right) d\lambda.$$

Then, by (23),

$$h(x) \leq \frac{1}{\pi}\int_{-\infty}^{\infty}(1 - \cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda < \infty, \quad \forall x \in \mathbb{R}. \tag{24}$$

This proves the finiteness of h. Now, using (22), we obtain

$$h_q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left(\frac{1 - e^{i\lambda x}}{q + \psi(\lambda)}\right) d\lambda.$$

Then, letting $q \to 0$ and using the dominated convergence theorem, (12) is obtained.

Note that

$$(1-\cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) \le 2(1\wedge\lambda^2) Re\left(\frac{1}{\psi(\lambda)}\right), \quad |x|\le 1, \quad \lambda\in\mathbb{R}.$$

Then, by (23) and dominated convergence theorem, it follows

$$\lim_{x\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos\lambda x) Re\left(\frac{1}{\psi(\lambda)}\right) d\lambda = 0.$$

Hence, by (24), $\lim_{x\to 0} h(x) = 0$. This proves that h is continuous at zero. Furthermore, since h is subadditive on $\mathbb{R}$, the continuity of h at the point zero implies the continuity on the whole real line (see e.g. [15, Theorem 6.8.2]).

Finally to prove that $x=0$ is the only point where h vanishes, we proceed by contradiction. Suppose that $h(x_0)=0$, for some $x_0\neq 0$. Using the subadditivity of h and making induction we get that $h(kx_0)=0$ for all $k\in\mathbb{Z}$. Besides, taking for granted the property $\lim_{|y|\to\infty} h(y) = \frac{1}{\kappa} > 0$, which will be proved in Lemma 15, the claim $h(kx_0)=0$, for all $k\in\mathbb{Z}$ is a contradiction. Therefore, $h(x)>0$, for all $x\neq 0$. □

3.3 Another representation for h_q and the behaviour of h at infinity

In this section we make the connection with the results from Section 23 in [18], but before we introduce further notation. For a Borel set B, let T_B be the first hitting time of B, that is, $T_B = \inf\{t>0 : X_t\in B\}$ (with $\inf\{\emptyset\}=\infty$). Let $(P_t^B, t\ge 0)$ be the semigroup of the Lévy process killed at T_B and $U_B(x,A)=\int_0^\infty \mathbb{P}_x(X_t\in A, t<T_B)dt$. For $f:\mathbb{R}\to\mathbb{R}$ an integrable Borel function, we denote

$$J(f)=\int_{\mathbb{R}} f(x)dx, \quad \widehat{f}(\lambda)=\int_{\mathbb{R}} f(x)e^{i\lambda x}dx, \quad \lambda\in\mathbb{R}.$$

Let $\mathfrak{F}^+$ be the class of non-negative, continuous, integrable functions f, whose Fourier transform has compact support and satisfies the following property: there exists a compact set K, a positive and finite constant c, and an open neighbourhood of zero V such that

$$J(f) - Re\widehat{f}(\lambda) \le c\max_{x\in K}(1-(2\pi)^{-1}\cos\lambda x), \quad \lambda\in V.$$

Let $\mathfrak{F}^*$ be the collection of differences of elements of $\mathfrak{F}^+$. Now, for $q>0$, let A_q and H_q^0 be given by

$$A_q f(x) = c_q J(f) - U_q f(x), \quad H_q^0 f(x) = f(0)\mathbb{E}_x(e^{-qT_0}), \quad x\in\mathbb{R}, \tag{25}$$

where c_q is a positive constant. As in [18], the constant c_q is taken to be equal to $U_q g(0)$, with g a symmetric function in $\mathfrak{F}^+$ satisfying $J(g)=1$.

It is said that a function f is essentially invariant if for each $t>0$, $f=P_t^B f$ a.e. Port and Stone proved that the only bounded essentially P_t^B-invariant functions are of the form $C\mathbb{P}_x(T_B=\infty)$ a.e., with C a positive constant (see Theorem 23.1). Furthermore, in the case X

recurrent, if B is such that $U_B(x, A)$ is bounded in x for any compact sets A, then the function $L_B(x)$ given by

$$L_B(x) = \lim_{q\to 0} qc_q \int_0^\infty \mathbb{P}_x(T_B > t)e^{-qt}dt, \quad x \in \mathbb{R}, \tag{26}$$

is a P_t^B-invariant function. Port and Stone proved furthermore that the constant c_q above introduced is such that for all $x \in \mathbb{R}$, the limit $\lim_{q\to 0} A_q f(x)$ exists, for $f \in \mathfrak{F}^*$.

The following lemma establishes an identity for h_q in terms of certain classes of functions. This identity is inspired from [18].

Lemma 13. *Let $f \in \mathfrak{F}^*$. Then,*

$$A_q f(x) - H_q^0 A_q f(x) = -U_q^0 f(x) + c_q(1 - H_q^0 \mathbf{1}(x))J(f), \tag{27}$$

where $\mathbf{1}$ denotes the constant function equal to 1. If $J(f) = 1$, the following holds for $x \in \mathbb{R}$,

$$A_q f(x) - A_q f(0)\mathbb{E}_x(e^{-qT_0}) = -U_q^0 f(x) + \frac{c_q}{u_q(0)} h_q(x). \tag{28}$$

Proof. By the strong Markov property, we have

$$\begin{aligned}
\mathbb{E}_x\left(\int_{T_0}^\infty e^{-qt} f(X_t)dt, T_0 < \infty\right) &= \mathbb{E}_x\left(e^{-qT_0}\int_0^\infty e^{-qu} f(X_{u+T_0})du, T_0 < \infty\right) \\
&= \mathbb{E}_x\left(e^{-qT_0}, T_0 < \infty\right)\mathbb{E}\left(\int_0^\infty e^{-qu} f(X_u)du\right) \\
&= H_q^0 \mathbf{1}(x) U_q f(0).
\end{aligned}$$

Thus,

$$\begin{aligned}
U_q f(x) &= \mathbb{E}_x\left(\int_0^{T_0} e^{-qt} f(X_t)dt\right) + \mathbb{E}_x\left(\int_{T_0}^\infty e^{-qt} f(X_t)dt, T_0 < \infty\right) \\
&= U_q^0 f(x) + U_q f(0) H_q^0 \mathbf{1}(x).
\end{aligned} \tag{29}$$

On the other hand, since $H_q^0 A_q f(x) = A_q f(0) H_q^0 \mathbf{1}(x)$, we have

$$H_q^0 A_q f(x) + U_q f(0) H_q^0 \mathbf{1}(x) = J(f) c_q H_q^0 \mathbf{1}(x). \tag{30}$$

Using (29), (30) and the definition of $A_q f$ we obtain

$$\begin{aligned}
A_q f(x) - H_q^0 A_q f(x) &= c_q J(f) - U_q^0 f(x) - U_q f(0) H_q^0 \mathbf{1}(x) - H_q^0 A_q f(x) \\
&= -U_q^0 f(x) + c_q(1 - H_q^0 \mathbf{1}(x))J(f),
\end{aligned}$$

which is (27).

Now, suppose that $J(f) = 1$. To obtain (28), we use the expression (25) and (27). $\square$

Remarks 14. (i) Let $\kappa = \lim_{q\to 0} \frac{1}{u_q(0)}$. From the identity,

$$\mathbb{E}_x[e^{-qT_0}] = 1 - \frac{1}{u_q(0)} h_q(x), \quad q > 0, \quad x \in \mathbb{R},$$

making $q \to 0$, it follows $\mathbb{P}_x(T_0 = \infty) = \kappa h(x)$, $x \in \mathbb{R}$.

(ii) In general by (19), we have that $h_q(x)$ can be written as

$$h_q(x) = u_q(0)\mathbb{P}_x(T_0 > \mathbf{e}_q) = u_q(0)\int_0^\infty \mathbb{P}_x(T_0 > t)qe^{-qt}dt, \quad q > 0, \quad x \in \mathbb{R}.$$

Letting $q \to 0$ and taking $B = \{0\}$ in (26), we obtain

$$h(x) = kL_{\{0\}}(x), \quad x \in \mathbb{R},$$

where $k = \lim_{q\to 0} \frac{u_q(0)}{c_q}$. Since $0 < h(x_0) < \infty$ and $0 < L_{\{0\}}(x_0) < \infty$, for some $x_0 \in \mathbb{R}$ (Lemma 2 and Theorem 18.3 in [18]), it follows that $0 < k < \infty$. Taking limit as $q \to 0$ in (28), we obtain

$$kh(x) = Af(x) - Af(0)\mathbb{P}_x(T_0 < \infty) + U^0 f(x), \quad x \in \mathbb{R}, \quad f \in \mathfrak{F}^*, \tag{31}$$

where $Af(x) = \lim_{q\to 0} A_q f(x)$ and $U^0 f(x) = \lim_{q\to 0} U_q^0 f(x)$, $x \in \mathbb{R}$.

To end this section, we establish the behaviour of h at infinity.

Lemma 15. *Let $\kappa := \lim_{q\to 0} \frac{1}{u_q(0)}$. We have the following*

(i) Suppose that either $\mathbb{E}(X_1^+) \leq \infty$ and $\mathbb{E}(X_1^-) < \infty$ or $\mathbb{E}(X_1^-) \leq \infty$ and $\mathbb{E}(X_1^+) < \infty$. If $0 < \mu := \mathbb{E}(X_1) \leq \infty$, then

$$\lim_{x\to\infty} h(x) = \frac{1}{\kappa}, \quad \lim_{x\to-\infty} h(x) = \frac{1}{\kappa} - \frac{1}{\mu};$$

while if $-\infty \leq \mu < 0$, then

$$\lim_{x\to\infty} h(x) = \frac{1}{\kappa} + \frac{1}{\mu}, \quad \lim_{x\to-\infty} h(x) = \frac{1}{\kappa}.$$

(ii) Suppose that X is recurrent, then

$$\lim_{|x|\to\infty} h(x) = \frac{1}{\kappa}.$$

Proof. We start by proving (i) in the case $0 < \mu \leq \infty$, the other case can be proved similarly. Set $f(x) = u_1(x)$, $x \in \mathbb{R}$. Note that $u_0(x) = \sum_{n=1}^\infty f^{*n}(x)$. Indeed,

$$\begin{aligned}
\sum_{n=1}^\infty f^{*n}(x)dx &= \sum_{n=1}^\infty \int_0^\infty \frac{s^{n-1}}{(n-1)!}e^{-s}\mathbb{P}(X_s \in dx) \\
&= \int_0^\infty e^{-s}\sum_{n=1}^\infty \frac{s^{n-1}}{(n-1)!}\mathbb{P}(X_s \in dx) \\
&= \int_0^\infty \mathbb{P}(X_s \in dx) \\
&= u_0(x)dx.
\end{aligned}$$

Furthermore, the Fourier transform of f is given by $\widehat{f}(\lambda) = 1/(1+\psi(\lambda))$, $\lambda \in \mathbb{R}$. Since $h(x) = u_0(0) - u_0(-x)$, it is suffices to compute the limit at infinity of $\sum_{n=1}^\infty f^{*n}(x)$. To that aim, we use the main result in [21], which states that if

(a) $\lim_{|x|\to\infty} f(x) = 0$,

(b) f is in $L_{1+\epsilon}$, for some $\epsilon > 0$,

then

$$\sum_{n=1}^{\infty} f^{*n}(x) \to \frac{1}{\mu}, \text{ as } x \to \infty, \quad \sum_{n=1}^{\infty} f^{*n}(x) \to 0, \text{ as } x \to -\infty.$$

The condition (a) is obtained from the Riemann-Lebesgue theorem. To show that (b) is satisfied we use the Pancherel's theorem (see [19, p. 186], [22, p. 202]). Thus, we will show that $\widehat{f}$ is in L_2. Thereby,

$$\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\lambda = \int_{\mathbb{R}} \frac{1}{|1+\psi(\lambda)|^2} d\lambda \leq \int_{\mathbb{R}} \frac{Re(1+\psi(\lambda))}{|1+\psi(\lambda)|^2} d\lambda = \int_{\mathbb{R}} Re\left(\frac{1}{1+\psi(\lambda)}\right) d\lambda < \infty.$$

This concludes the first part of the lemma.

Now, we prove the claim in (ii). Suppose that X is recurrent. By Theorem 17.10 in [18], $\lim_{x\to\infty} Af(x) = \infty$, for $f \in \mathfrak{F}^*$ and $J(f) > 0$. Hence, by (31), it follows $\lim_{x\to\infty} h(x) = \infty$. To obtain the behaviour of h at the opposite direction, we we consider the dual process $\widehat{X}$. For the dual process, we have that the invariant function $\widehat{h}$ for the semigroup $(\widehat{P}_t^0, t \geq 0)$ is given by $\widehat{h}(x) = \lim_{q\to 0}[\widehat{u}_q(0) - \widehat{u}_q(-x)] = \lim_{q\to 0}[u_q(0) - u_q(x)] = h(-x)$, $x \in \mathbb{R}$. The latter remark and the first part of the proof imply $\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} \widehat{h}(x) = \infty$. $\square$

3.4 An auxiliary function

Let $(h_q^*)_{q>0}$ be the increasing sequence of functions defined by

$$h_q^*(x) = \mathbb{E}\left(\int_0^{T_x} e^{-qt} dL_t\right), \quad q > 0, \quad x \in \mathbb{R},$$

where $T_x = \inf\{t > 0 : X_t = x\}$, the first hitting time of x for X. The sequence $(h_q^*)_{q>0}$ has the properties listed in the following proposition.

Proposition 16. *For any $q > 0$, the function h_q^* is a symmetric, nonnegative, subadditive continuous function, which can be expressed in terms of the q-resolvent density as*

$$h_q^*(x) = u_q(0) - \frac{u_q(x)u_q(-x)}{u_q(0)}, \quad x \in \mathbb{R}. \tag{32}$$

Proof. By definition, h_q^* is a non negative function. The continuity and symmetry of h_q^* is obtained from (32). Thus, it only remains to prove (32) and that h_q^* is subadditive.

First, we recall an expression that establishes a relation between resolvent densities and local times, (see Lemma 3 and commentary before Proposition 4 in [4, Chapter V]):

$$u_q(-x) = \mathbb{E}_x\left(\int_0^{\infty} e^{-qt} dL_t\right) = \mathbb{E}\left(\int_0^{\infty} e^{-qt} dL(x,t)\right), \quad q > 0, \quad x \in \mathbb{R}, \tag{33}$$

where $(L(x,t), t \geq 0)$ is the local time at point x for $(X, \mathbb{P})$. Thus, using the latter expression, we have

$$u_q(0) = \mathbb{E}\left(\int_0^\infty e^{-qt} dL_t\right) = h_q^*(x) + \mathbb{E}\left(\int_{T_x}^\infty e^{-qt} dL_t, T_x < \infty\right). \tag{34}$$

On the other hand, by Markov and additivity properties of local time, it follows

$$\begin{aligned}
\mathbb{E}\left(\int_{T_x}^\infty e^{-qt} dL_t, T_x < \infty\right) &= \mathbb{E}\left(e^{-qT_x} \int_0^\infty e^{-qu} dL_{u+T_x}, T_x < \infty\right) \\
&= \mathbb{E}(e^{-qT_x}, T_x < \infty)\mathbb{E}_x\left(\int_0^\infty e^{-qu} dL_u\right) \\
&= \widehat{\mathbb{E}}_x(e^{-qT_0}, T_0 < \infty)\mathbb{E}_x\left(\int_0^\infty e^{-qu} dL_u\right).
\end{aligned}$$

Then, using (4) and (33), the equation (34) becomes

$$u_q(0) = h_q^*(x) + \frac{u_q(x)}{u_q(0)} u_q(-x), \quad x \in \mathbb{R}.$$

Hence, (32) is obtained.

Now, we prove the subadditivity of h_q^*. The procedure is similar to the one used to prove the subadditivity of h_q in Lemma 11. We repeat the arguments for clarity. First, by (2) and (4) we can write (32) as

$$h_q^*(x) = u_q(0)(1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0})). \tag{35}$$

Since, for any $x \in \mathbb{R}$, $\mathbb{E}_x(e^{-qT_0}), \widehat{\mathbb{E}}_x(e^{-qT_0}) \leq 1$, it follows

$$(1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0}))(1 - \mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0})) \geq 0, \quad x, y \in \mathbb{R}.$$

The latter relation and (20) imply

$$\begin{aligned}
1 - \mathbb{E}_x(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0}) + 1 - \mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0}) &\geq 1 - \mathbb{E}_x(e^{-qT_0})\mathbb{E}_y(e^{-qT_0})\widehat{\mathbb{E}}_x(e^{-qT_0})\widehat{\mathbb{E}}_y(e^{-qT_0}) \\
&\geq 1 - \mathbb{E}_{x+y}(e^{-qT_0})\widehat{\mathbb{E}}_{x+y}(e^{-qT_0}),
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Hence, by (35)

$$h_q^*(x) + h_q^*(y) \geq h_q^*(x+y), \quad x, y \in \mathbb{R}.$$

This ends the proof. $\square$

Remark 17. With help of the expression (32), h_q^* can be written in terms of the function h_q as:

$$h_q^*(x) = h_q(x) + h_q(-x) - \frac{1}{u_q(0)} h_q(x) h_q(-x), \quad x \in \mathbb{R}. \tag{36}$$

Now, define h^* by

$$h^*(x) = \lim_{q \to 0} h_q^*(x), \quad x \in \mathbb{R}.$$

Since h is finite, then (36) implies that $h^*(x)$ is finite for all $x \in \mathbb{R}$. Furthermore, since

$$h_q^*(x) = \mathbb{E}\left(\int_0^{T_x} e^{-qt} dL_t\right) = \mathbb{E}\left(\int_0^{\infty} e^{-qs} \mathbf{1}_{\{X_u \neq -x, 0 \leq u \leq s\}} dL_s\right),$$

then

$$h^*(x) = \mathbb{E}(L_{T_x}) = \mathbb{E}\left(\int_0^{\infty} \mathbf{1}_{\{X_u \neq -x, 0 \leq u \leq s\}} dL_s\right), \quad x \in \mathbb{R}. \tag{37}$$

It is known that L_{T_x} is an exponential random variable. Thus, $h^*(x)$ is the expected value of an exponential random variable. We also note that by (36), in the recurrent symmetric case, h^* correspond to $2h^Y$, where h^Y is the invariant function given in [23].

Before we give some properties of the function h^*, we have the following technical lemma.

Lemma 18. *(i) For any $x \in \mathbb{R}$, $\lim_{q \to 0} q u_q(x) = 0$.*

(ii) For any $q, r > 0$, $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} u_q(y - x) u_r(y) dy = \frac{u_r(x) + u_q(-x)}{r + q}.$$

Proof. Recall the identity

$$\frac{u_q(x)}{u_q(0)} = \widehat{\mathbb{E}}_x(e^{-qT_0}), \quad x \in \mathbb{R}.$$

Hence, $q u_q(x) \sim \widehat{\mathbb{P}}_x(T_0 < \infty) q u_q(0)$ as $q \downarrow 0$. Thus, it is suffices to prove the case $x = 0$. Thanks to (22), we have

$$q u_q(0) = \frac{1}{2\pi} \int_{\mathbb{R}} Re\left(\frac{q}{q + \psi(\lambda)}\right) d\lambda.$$

For every $q > 0$, let j_q be the integrand function appearing in the latter display. Note that j_q can be expressed as

$$j_q(\lambda) = \left[1 + \frac{Re\psi(\lambda)}{q} + \frac{(Im\psi(\lambda))^2}{q(q + Re\psi(\lambda))}\right]^{-1}, \quad q > 0, \quad \lambda \in \mathbb{R}.$$

Hence, $j_q \downarrow 0$, as $q \downarrow 0$. Since $0 \leq j_q(\lambda) \leq [1 + \psi(\lambda)]^{-1}$, for all $\lambda \in \mathbb{R}$, and

$$\int_{\mathbb{R}} Re\left(\frac{1}{1 + \psi(\lambda)}\right) d\lambda < \infty,$$

the dominated convergence theorem implies $\lim_{q \to 0} q u_q(0) = 0$. This shows (i).

Now, let f be a positive, bounded, measurable function. We have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} u_q(y-x)u_r(y)dyf(x)dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y-z)u_q(z)dzu_r(y)dy \\ &= \frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r}-X_{\mathbf{e}_q})\right) \\ &= \frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r}-X_{\mathbf{e}_q})\mathbf{1}_{\{\mathbf{e}_r>\mathbf{e}_q\}}\right) \\ &\quad+\frac{1}{rq}\mathbb{E}\left(f(-(X_{\mathbf{e}_q}-X_{\mathbf{e}_r}))\mathbf{1}_{\{\mathbf{e}_q>\mathbf{e}_r\}}\right),\end{aligned}$$

where $\mathbf{e}_q$, $\mathbf{e}_r$ are independent exponential random variables with parameters $q>0$ and $r>0$, respectively, which are independent of $(X,\mathbb{P})$. The first term in the latter equation becomes

$$\begin{aligned}\frac{1}{rq}\mathbb{E}\left(f(X_{\mathbf{e}_r}-X_{\mathbf{e}_q})\mathbf{1}_{\{\mathbf{e}_r>\mathbf{e}_q\}}\right) &= \int_0^\infty\int_s^\infty e^{-rt}\mathbb{E}(f(X_t-X_s))dte^{-qs}ds \\ &= \int_0^\infty\int_s^\infty e^{-r(t-s)}\mathbb{E}(f(X_{t-s}))dte^{-(r+q)s}ds \\ &= \int_0^\infty e^{-(r+q)s}U_rf(0)ds \\ &= \frac{1}{r+q}U_rf(0).\end{aligned}$$

In the same way, it can be verified that

$$\frac{1}{rq}\mathbb{E}\left(f(-(X_{\mathbf{e}_q}-X_{\mathbf{e}_r}))\mathbf{1}_{\{\mathbf{e}_q>\mathbf{e}_r\}}\right)=\frac{1}{r+q}\widehat{U}_qf(0).$$

Thus, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u_q(y-x)u_r(y)dyf(x)dx=\int_{\mathbb{R}}\left(\frac{u_r(x)+u_q(-x)}{r+q}\right)f(x)dx,$$

for all positive, bounded, measurable function f. By the continuity of u_r and u_q, we conclude

$$\int_{\mathbb{R}} u_q(y-x)u_r(y)dy=\frac{u_r(x)+u_q(-x)}{r+q},$$

for any $q,r>0$, $x\in\mathbb{R}$. □

Some properties of the function h^* are summarized in the following lemma.

Lemma 19. *The function h^* is a symmetric, nonnegative, subadditive, continuous function which vanishes only at the point $x=0$ and $\lim_{|x|\to\infty}h^*(x)=\kappa^{-1}$. Furthermore, h^* is integrable with respect to semigroup of the process killed at T_0, i.e., $P_t^0h^*(x)<\infty$, for all $t>0$, $x\in\mathbb{R}$.*

Proof. From the definition of h_q^* and (32) the non negativity and symmetry of h^* follows. The subadditivity of h^* is obtained from subadditivity of the sequence $(h_q^*)_{q>0}$. Furthermore, once

we prove that h^* is subadditive and $h^*(x) \to \kappa^{-1}$, as $|x| \to \infty$, we can proceed as in the proof of Theorem 2 (ii) to obtain that the only point at which h^* vanishes is the point $x = 0$. We observe that from (36), we can write h^* in terms of h as

$$h^*(x) = h(x) + h(-x) - \kappa h(x)h(-x), \tag{38}$$

where $\kappa = \lim_{q\to 0} \frac{1}{u_q(0)}$. Hence, h^* is continuous. The property $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$ follows from Lemma 15 and (38).

Finally, we prove that h^* is P_t^0-integrable. For $x \in \mathbb{R}$, we write $\widehat{h}_q(x) = h_q(-x)$, $q > 0$, and $\widehat{h}(x) = \lim_{q\to 0} \widehat{h}_q(x)$. Let s be the function defined by $s(x) = h(x) + \widehat{h}(x)$. By (38), $h^*(x) \le s(x)$, $x \in \mathbb{R}$. Thus, it is suffices to show that s is P_t^0-integrable.

Now, by (1), the following identities hold for $0 < r < q$,

$$\begin{aligned} U_q h_r(x) &= \int_{\mathbb{R}} u_q(y-x)h_r(y)dy \\ &= \int_{\mathbb{R}} u_q(y-x)\{u_r(0) - u_r(-y)\}dy \\ &= \frac{u_r(0)}{q} - \int_{\mathbb{R}} u_q(y-x)u_r(-y)dy \\ &= \frac{u_r(0)}{q} - \frac{1}{q-r}\{u_r(-x) - u_q(-x)\} \\ &= \frac{h_r(x)}{q} - \frac{ru_r(-x)}{q(q-r)} + \frac{u_q(-x)}{q-r}. \end{aligned} \tag{39}$$

Thanks to Lemma 18 (i), $h_r(x) \to h(x)$ as $r \to 0$ and Fatou's lemma, we obtain

$$U_q h(x) \le \frac{h(x) + u_q(-x)}{q}, \quad q > 0, \quad x \in \mathbb{R}. \tag{40}$$

On the other hand, by Lemma 18 (ii), we have

$$\begin{aligned} U_q \widehat{h}_r(x) &= \int_{\mathbb{R}} u_q(y-x)(u_r(0) - u_r(y))dy \\ &= \frac{u_r(0)}{q} - \frac{u_r(x) + u_q(-x)}{r+q} \\ &= \frac{\widehat{h}_r(x)}{r+q} + \frac{ru_r(0)}{q(r+q)} - \frac{u_q(-x)}{r+q}. \end{aligned} \tag{41}$$

Using again Lemma 18 (i), $\widehat{h}_r(x) \to \widehat{h}(x)$ as $r \to 0$, and Fatou's lemma, it follows

$$U_q \widehat{h}(x) \le \frac{\widehat{h}(x) - u_q(-x)}{q}, \quad q > 0, \quad x \in \mathbb{R}. \tag{42}$$

Adding (40) and (42), we obtain that for any $q > 0$, $x \in \mathbb{R}$,

$$qU_q s(x) \le s(x). \tag{43}$$

Hence, the function s is P_t-integrable and therefore P_t^0-integrable. $\square$

Remarks 20. (i) From (43) it is deduced that the function s is excessive for the semigroup $(P_t, t \geq 0)$. Since s is a nonnegative function, then s is an excessive function for the semigroup $(P_t^0, t \geq 0)$.

(ii) By (19) and (35), we have $h_q(x) \leq h_q^*(x) \leq h^*(x) \leq s(x)$, for all $q > 0$, $x \in \mathbb{R}$. On the other hand, the Lemma 19 and its proof ensure that h^* satisfies

$$P_t^0 h^*(x) \leq s(x), \quad qU_q h^*(x) \leq s(x), \quad q > 0, \quad x \in \mathbb{R}. \tag{44}$$

These inequalities will be useful in the proofs of Lemma 21, assertion (iii) in Theorem 2 and Proposition 7 .

Lemma 21 ensure that the inequality obtained in (40) in fact is an equality. This result was established in [23] in the symmetric case.

Lemma 21. *For any $q > 0$, $x \in \mathbb{R}$,*

$$U_q h(x) = \frac{h(x) + u_q(-x)}{q}.$$

Proof. Remark 20 (ii) states that the function h^* satisfies $h_q(x) \leq h^*(x)$, $U_q h^*(x) < \infty$, for all $q > 0$, $x \in \mathbb{R}$. Then, by the dominated convergence theorem and (39), it follows

$$U_q h(x) = \lim_{r \to 0} U_q h_r(x) = \frac{h(x) + u_q(-x)}{q}.$$

□

4 Proofs of the main results

Proof of (iii) in Theorem 2. The first part of the proof is inspired in the proof of Lemma 1 in [8]. Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$ and independent of $(X, \mathbb{P})$. We claim that for $q > 0$, $x \in \mathbb{R}$, it holds,

$$\mathbb{E}_x(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{T_0 > t\}}) = e^{qt}\left(\mathbb{P}_x(T_0 > \mathbf{e}_q) - \int_0^t \mathbb{P}_x(T_0 > s)qe^{-qs}ds\right). \tag{45}$$

Indeed, by (21), we have

$$\mathbb{E}_x\left(\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)\mathbf{1}_{\{t < T_0\}}\right) = \mathbb{P}_x(T_0 > t + \mathbf{e}_q).$$

Now, making the change of variable $u = t + s$, we obtain

$$\begin{aligned}
\mathbb{P}_x(T_0 > t + \mathbf{e}_q) &= \int_0^\infty \mathbb{P}_x(T_0 > t + s)q^{-qs}ds \\
&= e^{qt}\int_t^\infty \mathbb{P}_x(T_0 > u)qe^{-qu}du \\
&= e^{qt}\left(\int_0^\infty \mathbb{P}_x(T_0 > u)qe^{-qu}du - \int_0^t \mathbb{P}_x(T_0 > u)qe^{-qu}du\right) \\
&= e^{qt}\left(\mathbb{P}_x(T_0 > \mathbf{e}_q) - \int_0^t \mathbb{P}_x(T_0 > u)qe^{-qu}du\right).
\end{aligned}$$

Hence, (45) follows.

By Remark 20 (ii) and Lemma 19, we have that the sequence $(h_q)_{q>0}$ is dominated by h^* and h^* is integrable with respect to P_t^0 for any $t > 0$. Then, using dominated convergence theorem, (19) and (45), it follows

$$\begin{aligned}
\mathbb{E}_x\left(h(X_t), t < T_0\right) &= \mathbb{E}_x\left(\lim_{q\to 0} h_q(X_t), t < T_0\right) \\
&= \lim_{q\to 0} \mathbb{E}_x\left(\frac{\mathbb{P}_{X_t}(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}\mathbf{1}_{\{t<T_0\}}\right) \\
&= \lim_{q\to 0} e^{qt}\left(\frac{\mathbb{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)} - \int_0^t \frac{\mathbb{P}_x(T_0 > u)}{n(\zeta > \mathbf{e}_q)} q e^{-qu} du\right) \\
&= h(x) - \frac{1}{n(\zeta)}\int_0^t \mathbb{P}_x(T_0 > u) du,
\end{aligned}$$

where $n(\zeta) = \lim_{q\to 0}\int_0^\infty e^{-qt} n(\zeta > t)dt$. On the other hand, Lemma 18 and (7) imply $n(\zeta) = \lim_{q\to 0}[qu_q(0)]^{-1} = \infty$. Therefore, we conclude

$$\mathbb{E}_x\left(h(X_t), t < T_0\right) = h(x), \quad t > 0, \quad x \in \mathbb{R}.$$

Now, we prove the second part of (iii) in Theorem 2. From (6) and Lemma 21, we obtain that the Laplace transform of $n(h(X_t), t < \zeta)$ is given by

$$\int_0^\infty e^{-qt} n(h(X_t), t < \zeta)dt = \int_{\mathbb{R}} h(x)\widehat{\mathbb{E}}_x[e^{-qT_0}]dx = \int_{\mathbb{R}} h(x)\frac{u_q(x)}{u_q(0)}dx = \frac{1}{u_q(0)}U_q h(0) = \frac{1}{q}.$$

Hence, the claim follows. $\square$

Proof of Theorem 5. The only thing which has to be proved is the fact that $\mathbb{P}_0^\updownarrow$ is a Markovian probability measure with the same semigroup as under $\mathbb{P}_x^\updownarrow$, $x \neq 0$ and that $\mathbb{P}_0^\updownarrow(X_0 = 0) = 1$. Since n is a Markovian measure (σ-finite) with semigroup $(P_t^0, t \geq 0)$. Let g be bounded Borel function and $\Lambda \in \mathcal{F}_t$ and $t, s > 0$:

$$\begin{aligned}
\mathbb{E}_0^\updownarrow(\mathbf{1}_\Lambda g(X_{t+s})) &= n(\mathbf{1}_\Lambda h(X_{t+s})g(X_{t+s})\mathbf{1}_{\{t+s<\zeta\}}) \\
&= n(\mathbf{1}_\Lambda \mathbb{E}_{X_t}^0(h(X_s)g(X_s))\mathbf{1}_{\{t<\zeta\}}) \\
&= n(\mathbf{1}_\Lambda h(X_t)\mathbb{E}_{X_t}^\updownarrow(h(X_s)g(X_s))\mathbf{1}_{\{t<\zeta\}}) \\
&= \mathbb{E}_0^\updownarrow(\mathbf{1}_\Lambda \mathbb{E}_{X_t}^\updownarrow(g(X_s))).
\end{aligned}$$

This shows the first part. Now, we prove that $\mathbb{P}_0^\updownarrow(X_0 = 0) = 1$. Since X is right continuous at 0, it is sufficient to prove that for any $z > 0$,

$$\mathbb{P}_0^\updownarrow(|X_\epsilon| < z) \to 1,$$

as $\epsilon \to 0$. The latter is equivalent to prove

$$\lim_{\epsilon\to 0} n(\mathbf{1}_{\{|X_\epsilon|>z\}}h(X_\epsilon)\mathbf{1}_{\{\epsilon<\zeta\}}) = 0.$$

Since $n(h(X_s), s < \zeta) = 1$, $\mathcal{Q}_s(\cdot) := n(\cdot, h(X_s), s < \zeta)$ defines a probability measure. Then, from the Markov property, for all $\epsilon < s$, $\mathbb{P}_0^{\updownarrow}(|X_\epsilon| < z) = \mathcal{Q}_s(\mathbf{1}_{\{|X_\epsilon|<z\}})$. Since the excursions of the Lévy process $(X, \mathbb{P})$ leave 0 continuously, we have $\mathbf{1}_{\{|X_\epsilon|<z\}} \to 1$, $\mathcal{Q}_s$-a.s. as $\epsilon \to 0$. The result follows from the dominated converge theorem. □

Proof of Theorem 6. We proceed as in [8]. Let $x \neq 0$, T a $(\mathcal{F}_t)_{t\geq 0}$ stopping time and $\Lambda \in \mathcal{F}_T$. With the help of the strong Markov property and since $\mathbf{e}_q$ is independent of $(X, \mathbb{P})$, we can deduce the following

$$\begin{aligned}
\mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<\mathbf{e}_q\}}\mathbf{1}_{\{T_0>\mathbf{e}_q\}}\right) &= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\}}\mathbf{1}_{\{T<s\}}\mathbf{1}_{\{T_0>s\}}\right) qe^{-qs}ds \\
&= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\}}\mathbf{1}_{\{T<s\}}\mathbb{E}_x\left(\mathbf{1}_{\{T_0>s\}}\circ\theta_T \mid \mathcal{F}_T\right)\right) qe^{-qs}ds \\
&= \int_0^\infty \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\wedge s\}}\mathbb{P}_{X_T}(T_0>s)\right) qe^{-qs}ds \\
&= \mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\wedge \mathbf{e}_q\}}\mathbb{P}_{X_T}(T_0>\mathbf{e}_q)\right) \\
&= n(\zeta > \mathbf{e}_q)\mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\wedge \mathbf{e}_q\}}h_q(X_T)\right) \\
&= \frac{1}{h_q(x)}\mathbb{E}_x\left(\mathbf{1}_\Lambda \mathbf{1}_{\{T<T_0\wedge \mathbf{e}_q\}}h_q(X_T)\right)\mathbb{P}_x(T_0>\mathbf{e}_q).
\end{aligned}$$

The latter shows that for $\Lambda \in \mathcal{F}_T$, T stopping time finite a.s.

$$\mathbb{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \frac{1}{h_q(x)}\mathbb{E}_x\left(\mathbf{1}_\Lambda h_q(X_T)\mathbf{1}_{\{T<T_0\wedge\mathbf{e}_q\}}\right). \tag{46}$$

Now, recall that $h_q(x) \leq h_q^*(x) \leq h^*(x)$, $q > 0$, $x \in \mathbb{R}$. Thus,

$$\mathbf{1}_{\{T<T_0\wedge\mathbf{e}_q\}}h_q(X_T) \leq \mathbf{1}_{\{T<T_0\}}h^*(X_T) \quad \text{a.s.}$$

On the other hand, the first inequality in (44) also it is satisfied for stopping times, i.e., $\mathbb{E}_x(h^*(X_T), T < T_0) \leq s(x)$. Then, letting $q \to 0$, with the help of the dominated convergence theorem in (46), we obtain the desired result. □

Proof of Proposition 7. For every $s > 0$, we consider $d_s = \inf\{u > s : X_u = 0\}$, $g_s = \sup\{u \leq s : X_u = 0\}$ and $G = \{g_u : g_u \neq d_u, u > 0\}$. By definition, for every $q > 0$, $\Lambda \in \mathcal{F}_t$, we have

$$\begin{aligned}
\mathbb{P}^{\mathbf{e}_q}(\Lambda, t<\zeta) &= \mathbb{E}(\mathbf{1}_\Lambda \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}} \circ \theta_{g_{\mathbf{e}_q}}\mathbf{1}_{\{t<\mathbf{e}_q - g_{\mathbf{e}_q}\}}) \\
&= \mathbb{E}\left(\int_0^\infty \mathbf{1}_\Lambda \circ k_{u-g_u}\circ\theta_{g_u}\mathbf{1}_{\{t<u-g_u\}}qe^{-qu}du\right) \\
&= \mathbb{E}\left(\sum_{s\in G} e^{-qs}\int_s^{d_s} qe^{-q(u-s)}\mathbf{1}_\Lambda\circ k_{u-s}\circ\theta_s\mathbf{1}_{\{t<u-s\}}du\right).
\end{aligned}$$

Now, using the compensation formula in excursion theory (see e.g. [4], [17]) and the strong Markov property of n, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sum_{s\in G} e^{-qs}\int_s^{d_s} qe^{-q(u-s)}\mathbf{1}_\Lambda\circ k_{u-s}\circ\theta_s\mathbf{1}_{\{t<u-s\}}du\right) &= \mathbb{E}\left(\int_0^\infty e^{-qs}dL_s\right)n(\mathbf{1}_\Lambda\mathbf{1}_{\{t<\mathbf{e}_q<\zeta\}}) \\
&= \mathbb{E}\left(\int_0^\infty e^{-qs}dL_s\right)n(\mathbf{1}_\Lambda\mathbb{P}_{X_t}(T_0>\mathbf{e}_q)\mathbf{1}_{\{t<\zeta\}}).
\end{aligned}$$

Using (7) and (34) we deduce

$$\mathbb{E}\left(\int_0^\infty e^{-qs} dL_s\right) = u_q(0) = \frac{1}{n(\zeta > \mathbf{e}_q)}.$$

Thus, we see that

$$\mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = n(\mathbf{1}_\Lambda h_q(X_t)\mathbf{1}_{\{t<\zeta\}}). \tag{47}$$

Now, we prove that $n(s(X_t), t < \zeta) < \infty$, for all $t > 0$. First, note that since s is excessive for the semigroup $(P_t^0, t \geq 0)$ and n fulfils the Markov property, then $t \mapsto n(s(X_t), t < \zeta)$ is decreasing. This is verified from the following equalities: for $u, t > 0$,

$$\begin{aligned} n(s(X_{t+u}), t+u < \zeta) &= n((s(X_u)\mathbf{1}_{\{u<\zeta\}}) \circ \theta_t, t < \zeta) \\ &= n(\mathbb{E}_{X_t}(s(X_u), u < \zeta), t < \zeta) \\ &= n(P_u^0 s(X_t), t < \zeta) \\ &\leq n(s(X_t), t < \zeta). \end{aligned}$$

On the other hand, by (6) and remark 20 (i), we have

$$\int_0^\infty e^{-t} n(s(X_t), t < \zeta) dt = \int_{\mathbb{R}} s(x) \frac{u_1(x)}{u_1(0)} dx = \frac{1}{u_1(0)} U_1 s(0) \leq \frac{1}{u_1(0)} s(0).$$

Therefore, $n(s(X_t), t < \zeta)$ is finite for every $t > 0$.

Finally, since $\mathbf{1}_\Lambda h_q(X_t) \leq s(X_t)$ and $n(s(X_t), t < \zeta) < \infty$, we can apply the dominated convergence theorem in (47) to conclude that for $t > 0$ fixed

$$\lim_{q \to 0} \mathbb{P}^{\mathbf{e}_q}(\Lambda, t < \zeta) = n(\mathbf{1}_\Lambda h(X_t)\mathbf{1}_{\{t<\zeta\}}).$$

□

Let $U_q^\updownarrow$ be the q-resolvent for the process $X^\updownarrow = (X, \mathbb{P}_x^\updownarrow)_{x\in\mathbb{R}}$, with $U^\updownarrow = U_0^\updownarrow$. To prove that $X^\updownarrow$ is transient, we compute the density of $U^\updownarrow$. For $x, y \neq 0$ and $q > 0$, we have

$$u_q^\updownarrow(x, y) = \frac{h(y)}{h(x)} u_q^0(x, y). \tag{48}$$

From (6) it can be deduced that for $y \neq 0$, $q > 0$,

$$\begin{aligned} u_q^\updownarrow(0, y) dy &= \int_0^\infty e^{-qt} n(h(X_t)\mathbf{1}_{\{X_t \in dy\}}, t < \zeta) dt \\ &= h(y) \widehat{\mathbb{E}}_y[e^{-qT_0}] dy \\ &= h(y) \frac{u_q(y)}{u_q(0)} dy. \end{aligned} \tag{49}$$

Finally, by Theorem 5 (ii), $u_q^\updownarrow(x, 0) = 0$, for all x. Thus, from the above equations, the density of $U^\updownarrow$ can be obtained . This is stated in the following lemma.

Lemma 22. *Let* $u_0^{\updownarrow}(x,y) = \lim_{q\to 0} u_q^{\updownarrow}(x,y)$, $x, y \in \mathbb{R}$. *Then* $u_0^{\updownarrow}(x,0) = 0$, *for all* x,

$$0 \le u_0^{\updownarrow}(x,y) = \frac{h(y)}{h(x)}[h(x) + h(-y) - h(x-y) - \kappa h(x)h(-y)], \quad x \neq 0, y \neq 0, \tag{50}$$

and for $y \neq 0$,

$$u_0^{\updownarrow}(0,y) = h(y)(1 - \kappa h(-y)) = h^*(y) - h(-y). \tag{51}$$

Proof. An easy computation gives

$$\frac{u_q(-x)u_q(y)}{u_q(0)} = \frac{h_q(x)h_q(-y)}{u_q(0)} - h_q(x) - h_q(-y) + u_q(0), \quad x \neq 0, y \neq 0.$$

Using this and (3) it follows

$$u_q^0(x,y) = h_q(x) + h_q(-y) - h_q(x-y) - \frac{h_q(x)h_q(-y)}{u_q(0)}, \quad x \neq 0, y \neq 0. \tag{52}$$

Letting $q \to 0$ in (48) we obtain (50). The first equality in (51) is obtained from (49) recalling that for all y, $\lim_{q\to 0}[u_q(y)/u_q(0)] = \lim_{q\to 0}[1 - (u_q(0))^{-1}h_q(-y)] = 1 - \kappa h(-y)$. The second one follows from (38). $\square$

Remark 23. Note that from (36) and (52) we have $u_q^{\updownarrow}(x,x) = u_q^0(x,x) = h_q^*(x)$, $x \neq 0$, which implies $u_0^{\updownarrow}(x,x) = h^*(x)$, $x \neq 0$.

Proof of Proposition 8. To obtain the transiency property of $X^{\updownarrow}$, we use Theorem 3.7.2 in [12], which states the following. If the conditions:

(i) $U^{\updownarrow}g$ is lower semi-continuous, for any non negative function g with compact support;

(ii) there exists a non negative function f such that $0 < U^{\updownarrow}f < \infty$ on $\mathbb{R}$;

are satisfied, then the process $X^{\updownarrow}$ is transient.

Since h is continuous, from Lemma 22 it follows $\lim_{x\to x'} u_0^{\updownarrow}(x,y) = u_0^{\updownarrow}(x',y)$, for all $y \in \mathbb{R}$. Let g be a non negative function with compact support K. By Fatou's lemma, we have

$$\liminf_{x\to x'} \int_K g(y)u_0^{\updownarrow}(x,y)dy \ge \int_{\mathbb{R}} g(y) \liminf_{x\to x'}[u_0^{\updownarrow}(x,y)\mathbf{1}_K]dy = \int_K g(y)u_0^{\updownarrow}(x',y)dy.$$

This shows that for any g non negative with compact support, the function

$$x \longmapsto \int_{\mathbb{R}} g(y)u_0^{\updownarrow}(x,y)dy$$

is lower semi-continuous. Thus, condition (i) is satisfied.

Now, we will find a non negative function $f : \mathbb{R} \to \mathbb{R}^+$ such that $0 < U^{\updownarrow}f(x) < \infty$. Let f be given by

$$f(y) = \begin{cases} \dfrac{1}{[h^*(1)]^2}, & |y| \le 1, \\[2ex] \dfrac{1}{y^2[h^*(y)]^2}, & |y| > 1. \end{cases}$$

Since f is continuous and $\lim_{|x|\to\infty} h^*(x) = \kappa^{-1}$, then f, fh^* and $f(h^*)^2$ are integrable with respect to Lebesgue measure. On the other hand, h is dominated by the symmetric function h^*, then the integrability of fh^* and $f(h^*)^2$ imply

$$\int_{\mathbb{R}} f(y)h(y)dy < \infty, \quad \int_{\mathbb{R}} f(y)h(y)h(-y)dy < \infty.$$

Furthermore, since h is subadditive and f is symmetric, it follows,

$$\int_{\mathbb{R}} f(y)h(x-y)dy \leq \int_{\mathbb{R}} f(y)h(x)dy + \int_{\mathbb{R}} f(y)h(y)dy < \infty.$$

Thus, for $x \neq 0$,

$$U^{\updownarrow} f(x) = \int_{\mathbb{R}} f(y)u_0^{\updownarrow}(x,y)dy < \infty.$$

Finally,

$$U^{\updownarrow} f(0) = \int_{\mathbb{R}} f(y)u^{\updownarrow}(0,y)dy = \int_{\mathbb{R}} f(y)(h^*(y) - h(y))dy < \infty.$$

This concludes the proof. $\square$

The Lemma 24 below states that any $x \neq 0$ is regular for itself under $\mathbb{P}_x^{\updownarrow}$. The latter implies the existence of a continuous local time at point x for the process $(X, \mathbb{P}_x^{\updownarrow})$, see [5, Theorem 3,12, p. 216]. We will denote by $(L^{\updownarrow}(x,t), t \geq 0)$ the local time at point x aforementioned and by $\tau^{\updownarrow}(x,t)$ the right continuous inverse of $L^{\updownarrow}(x,t)$, i.e.,

$$\tau^{\updownarrow}(x,t) = \inf\{s > 0 : L^{\updownarrow}(x,s) > t\}, \quad t \geq 0.$$

It is well known that $(\tau^{\updownarrow}(x,t), t \geq 0)$ is a subordinator killed at an exponential random time independent of $\tau^{\updownarrow}(x,\cdot)$ with Laplace exponent $\Phi^{x,\updownarrow}$ satisfying

$$\mathbb{E}_x(e^{-q\tau^{\updownarrow}(x,t)}) = e^{-t\Phi^{x,\updownarrow}(q)} = e^{-t/u_q^{\updownarrow}(x,x)}, \quad t \geq 0, \tag{53}$$

see e.g. [5, Theorem 3.17, p. 218]. Furthermore, using the compensation formula in excursion theory we can be establish that for any $q > 0$,

$$\begin{aligned}\Phi^{x,\updownarrow}(q) &= \frac{1}{u_q^{\updownarrow}(x,x)} = n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q \\ &= n_x^{\updownarrow}(\zeta = \infty) + a^x q + \int_0^\infty (1 - e^{-qt}) n_x^{\updownarrow}(\zeta \in dt),\end{aligned} \tag{54}$$

where a^x satisfies

$$\int_0^t \mathbf{1}_{\{X_s = x\}} ds = a^x L^{\updownarrow}(x,t). \tag{55}$$

By Remark 23, $\lim_{q\to 0} u_q^{\updownarrow}(x,x) = h^*(x) > 0$, for $x \neq 0$, then $(\tau^{\updownarrow}(x,t), t \geq 0)$ is a subordinator killed at an exponential time with parameter $1/h^*(x) > 0$. This also confirms the transiency of $(X, \mathbb{P}_x^{\updownarrow})$, since by (54), there exists an excursion of infinite length.

To state the following lemma, we introduce additional notation. For every $x \in \mathbb{R}$, define $d_s^x = \inf\{u > s : X_t = x\}$, $g_s^x = \sup\{u \leq s : X_t = x\}$ and $G^x = \{g_u^x : g_u^x \neq d_u^x, u > 0\}$.

Lemma 24. *(i) For $x \in \mathbb{R} \setminus \{0\}$, x is regular for itself for $(X, \mathbb{P}_x^{\updownarrow})$.*

(ii) Let $\mathbf{e}_q$ be an exponential random variable with parameter $q > 0$, independent of $(X, (\mathbb{P}_x^{\updownarrow})_{x \neq 0})$. Then, for every $x \neq 0$, the processes $(X_u, u < g_{\mathbf{e}_q}^x)$ and $X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x}$ are $\mathbb{P}_x^{\updownarrow}$ independent. Furthermore, their laws are characterized as follows: let F and H be measurable and bounded functionals, then

$$\mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g_{\mathbf{e}_q}^x)\right) = \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^{\updownarrow}(x, s)\right)\left[n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q\right] \quad (56)$$

and

$$\mathbb{E}_x^{\updownarrow}\left(H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x})\right) = u_q^{\updownarrow}(x, x)\left[n_x^{\updownarrow}\left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt\right) + a^x q H(\bar{x})\right], \quad (57)$$

where a^x is the constant in (55).

Proof. Let $x \in \mathbb{R} \setminus \{0\}$. By Fatou's lemma and the definition of $\mathbb{P}_x^{\updownarrow}$, we have

$$\begin{aligned}
\mathbb{P}_x^{\updownarrow}(T_x = 0) &= \liminf_{t \to 0} \mathbb{P}_x^{\updownarrow}(T_x \leq t) \\
&\geq \frac{1}{h(x)} \mathbb{E}_x\left(\liminf_{t \to 0} \mathbf{1}_{\{T_x \leq t < T_0\}} h(X_t)\right) \\
&= \frac{1}{h(x)} \mathbb{E}_x\left(\mathbf{1}_{\{T_x = 0\}} \mathbf{1}_{\{T_0 > 0\}} h(X_0)\right) \\
&= 1,
\end{aligned}$$

where the latter equality was obtained using the facts that $\{x\}$ is regular for itself under $\mathbb{P}_x$ and $\mathbb{P}_x(T_0 > 0) = 1$. This proves (i).

Before to prove (ii), we recall the following. Since $\tau^{\updownarrow}(x, \cdot)$ is the inverse of the local time $(L^{\updownarrow}(x, t), t \geq 0)$ with Laplace exponent given by (53), then

$$\mathbb{E}_x^{\updownarrow}\left(\int_0^\infty e^{-qt} dL^{\updownarrow}(x, t)\right) = \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty e^{-q\tau^{\updownarrow}(x,t)} dt\right) = \int_0^\infty \mathbb{E}_x^{\updownarrow}(e^{-q\tau^{\updownarrow}(x,t)}) dt = u_q^{\updownarrow}(x, x). \quad (58)$$

We will denote $\bar{x}$ the path which is identically equal to x and with lifetime zero. Thus, for F and H measurable and bounded functionals, using the compensation formula in excursion theory (see e.g. [4], [17]), it follows

$$\begin{aligned}
&\mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g_{\mathbf{e}_q}^x) H(X \circ k_{\mathbf{e}_q - g_{\mathbf{e}_q}^x} \circ \theta_{g_{\mathbf{e}_q}^x})\right) \\
&= \mathbb{E}_x^{\updownarrow}\left(\sum_{s \in G^x} F(X_u, u < s) e^{-qs} \int_s^{d_s} H(X \circ k_{t-s} \circ \theta_s) q e^{-q(t-s)} dt\right) \\
&\quad + \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty F(X_u, u < t) H(\bar{x}) q e^{-qt} \mathbf{1}_{\{X_t = x\}} dt\right) \\
&= \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^{\updownarrow}(x, s)\right)\left[n_x^{\updownarrow}\left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt\right) + a^x q H(\bar{x})\right], \quad (59)
\end{aligned}$$

where a^x is the constant in (55). Taking $H \equiv 1$ in (59), it follows

$$\mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g^x_{\mathbf{e}_q})\right) = \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty F(X_u, u < s) e^{-qs} dL^{\updownarrow}(x, s)\right)\left[n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q\right].$$

In the same way, if we take $F \equiv 1$ in (59) and we use (58), we can obtain

$$\mathbb{E}_x^{\updownarrow}\left(H(X \circ k_{\mathbf{e}_q - g^x_{\mathbf{e}_q}} \circ \theta_{g^x_{\mathbf{e}_q}})\right) = u_q^{\updownarrow}(x, x)\left[n_x^{\updownarrow}\left(\int_0^\zeta H(\epsilon_u, u < t) q e^{-qt} dt\right) + a^x q H(\bar{x})\right].$$

The latter two displays are (56) and (57), respectively.

Finally, by (54), $u_q^{\updownarrow}(x, x) = [n_x^{\updownarrow}(\zeta > \mathbf{e}_q) + a^x q]^{-1}$. Using this fact, (56) and (57), we conclude

$$\mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g^x_{\mathbf{e}_q}) H(X \circ k_{\mathbf{e}_q - g^x_{\mathbf{e}_q}} \circ \theta_{g^x_{\mathbf{e}_q}})\right) = \mathbb{E}_x^{\updownarrow}\left(F(X_u, u < g^x_{\mathbf{e}_q})\right)\mathbb{E}_x^{\updownarrow}\left(H(X \circ k_{\mathbf{e}_q - g^x_{\mathbf{e}_q}} \circ \theta_{g^x_{\mathbf{e}_q}})\right).$$

This shows the independence property in (ii). □

Now, we will prove that the drift coefficient in (54) does not depend on x, and is equal to δ.

Lemma 25. *Let δ be the drift coefficient of the inverse local time at the point zero for the Lévy process $(X, \mathbb{P})$. Then for all $x \in \mathbb{R} \setminus \{0\}$, $\mathbb{P}_x^{\updownarrow}$-a.s., $\int_0^t \mathbf{1}_{\{X_s = 0\}} ds = \delta L^{\updownarrow}(x, t)$. That is, $a^x = \delta$, for all $x \in \mathbb{R} \setminus \{0\}$.*

Proof. If both δ, a^x are zero, the claim holds. Suppose that $a^x \neq 0$. Using (58), the definition of a^x and $\mathbb{P}_x^{\updownarrow}$, we obtain

$$\begin{aligned} a^x u_q^{\updownarrow}(x, x) &= \mathbb{E}_x^{\updownarrow}\left(\int_0^\infty e^{-qt} d[a^x L^{\updownarrow}(x, t)]\right) \\ &= \int_0^\infty \mathbb{E}_x^{\updownarrow}\left(\mathbf{1}_{\{X_t = x\}}\right) e^{-qt} dt \\ &= \int_0^\infty \frac{1}{h(x)} \mathbb{E}_x\left(\mathbf{1}_{\{X_t = x\}} h(X_t) \mathbf{1}_{\{T_0 > t\}}\right) e^{-qt} dt \\ &= \mathbb{E}_x\left(\int_0^\infty \mathbf{1}_{\{T_0 > t\}} e^{-qt} \mathbf{1}_{\{X_t = x\}} dt\right). \end{aligned} \tag{60}$$

Using that $(X, \mathbb{P}_x)$ is equal in distribution to $(X + x, \mathbb{P})$, the definition of δ and the symmetry of $h_q^*(x)$, it follows that the right-hand side in (60) is

$$\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}} > t\}} e^{-qt} \mathbf{1}_{\{X_t = 0\}} dt\right) = \delta \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}} > t\}} e^{-qt} dL_t\right) = \delta h_q^*(x).$$

To conclude the proof recall that $h_q^*(x) = u_q^{\updownarrow}(x, x)$. □

Proof of Proposition 9. Let $H : \mathcal{D}^0 \to \mathbb{R}$ a bounded and measurable functional. To simplify we write X^q for the path $X \circ k_{\mathbf{e}_q - g^x_{\mathbf{e}_q}} \circ \theta_{g^x_{\mathbf{e}_q}}$. Using the definition of $\mathbb{P}_x^{\updownarrow}$, we obtain

$$h(x) \mathbb{E}_x^{\updownarrow}\left(H(X^q)\right) = \mathbb{E}\left(\int_0^\infty H((X + x) \circ k_{t - g_t} \circ \theta_{g_t}) h(X_t + x) \mathbf{1}_{\{T_{\{-x\}} > t\}} q e^{-qt} dt\right). \tag{61}$$

We note that $\mathbf{1}_{\{T_{\{-x\}}>t\}} = \mathbf{1}_{\{T_{\{-x\}}\circ\theta_{g_t}>t-g_t\}}\mathbf{1}_{\{T_{\{-x\}}>g_t\}}$ and $h(X_t+x) = h((X_{t-g_t}+x)\circ\theta_{g_t})$. Then, with the help of the compensation formula in excursion theory ([4], [17]), the right-hand side in (61) can be written as

$$\begin{aligned}
&\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>g_t\}} H((X+x)\circ k_{t-g_t}\circ\theta_{g_t})h((X_{t-g_t}+x)\circ\theta_{g_t})\mathbf{1}_{\{T_{\{-x\}}\circ\theta_{g_t}>t-g_t\}}qe^{-qt}dt\right)\\
&\quad= \mathbb{E}\left(\sum_{s\in G}\mathbf{1}_{\{T_{\{-x\}}>s\}}\int_s^{d_s} H((X+x)\circ k_{t-s}\circ\theta_s)h((X_{t-s}+x)\circ\theta_s)\mathbf{1}_{\{T_{\{-x\}}\circ\theta_s>t-s\}}qe^{-qt}dt\right)\\
&\qquad+\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}}H(\bar{x})h(x)qe^{-qt}\mathbf{1}_{\{X_t=0\}}dt\right)\\
&\quad= \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}}e^{-qt}dL_t\right)n\left(\int_0^\zeta H(\epsilon_u+x, u<t)h(X_t+x)\mathbf{1}_{\{T_{\{-x\}}>t\}}qe^{-qt}dt\right)\\
&\qquad+q\delta H(\bar{x})h(x)\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{T_{\{-x\}}>t\}}e^{-qt}dL_t\right)\\
&\quad= h_q^*(x)\left[n\left(\int_0^\zeta H(\epsilon_u+x,u<t)h(X_t+x)\mathbf{1}_{\{T_{\{-x\}}>t\}}qe^{-qt}dt\right)+q\delta H(\bar{x})h(x)\right],
\end{aligned}\tag{62}$$

where δ is such that $\int_0^t \mathbf{1}_{\{X_s=0\}} = \delta L_t$ under $\mathbb{P}$. Using Lemma 25 and $h^*(x) = u_q^{\updownarrow}(x,x)$ in (62), we verify

$$\mathbb{E}_x^{\updownarrow}(H(X^q)) = u_q^{\updownarrow}(x,x)\left[\frac{1}{h(x)}n\left(\int_0^\zeta H(\epsilon_u+x,u<t)h(X_t+x)\mathbf{1}_{\{T_{\{-x\}}>t\}}qe^{-qt}dt\right)+a^x qH(\bar{x})\right].\tag{63}$$

Comparing (63) with (57), the result follows. □

5 An example

The expression (12) in Lemma 2 (i) allows us to compute explicitly the function h in the particular case when $(X,\mathbb{P})$ is an α-stable process.

Example 26. Suppose that $(X,\mathbb{P})$ is an α-stable process. Then, $(X,\mathbb{P})$ satisfies **H.1** and **H.2** if and only if $\alpha\in(1,2]$. It is well known that the resolvent density of Brownian motion is $u_q(x) = (\sqrt{2q})^{-1}e^{-\sqrt{2q}|x|}$, hence $h(x) = \lim_{q\to 0}[u_q(0)-u_q(-x)] = |x|$. Now, let $\alpha\in(1,2)$. In this case the function h takes the following form

$$h(x) = K(\alpha)(1-\beta\,\mathrm{sgn}(x))|x|^{\alpha-1},$$

where

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\alpha\pi/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\alpha\pi/2))}$$

and

$$c = -\frac{(c^+ + c^-)\Gamma(2-\alpha)}{\alpha(\alpha-1)}\cos(\alpha\pi/2),\quad \beta = \frac{c^+ - c^-}{c^+ + c^-}.\tag{64}$$

Indeed, recall that the characteristic exponent of $(X, \mathbb{P})$ can be written as

$$\psi(\lambda) = c|\lambda|^\alpha(1 - i\beta \text{sgn}(\lambda)\tan(\alpha\pi/2)),$$

where c and β are as in (64). Now, we have

$$Re\left(\frac{1 - e^{i\lambda x}}{\psi(\lambda)}\right) = \frac{1 - \cos(\lambda x) + \beta\tan(\alpha\pi/2)\text{sgn}(\lambda)\sin(\lambda x)}{c|\lambda|^\alpha(1 + \beta^2\tan^2(\alpha\pi/2))}.$$

Since the function $\text{sgn}(\lambda)\sin(\lambda x)$ as function of λ is even, we have

$$h(x) = \frac{1}{c(1 + \beta^2\tan^2(\alpha\pi/2))}\left(h^s(x) + \beta\tan(\alpha\pi/2)\frac{1}{\pi}\int_0^\infty \frac{\sin(\lambda x)}{\lambda^\alpha}d\lambda\right),$$

where $h^s(x)$ is the function h obtained in the symmetric case (see example 1.1 in [23]), namely

$$h^s(x) = \frac{\Gamma(2-\alpha)}{\pi(\alpha - 1)}\sin(\alpha\pi/2)|x|^{\alpha-1}.$$

On the other hand, with the help of formulae (14.18) of Lemma 14.11 in [20], we obtain

$$\int_0^\infty \frac{\sin(\lambda x)}{\lambda^\alpha}d\lambda = \text{sgn}(x)|x|^{\alpha-1}\int_0^\infty \frac{\sin u}{u^\alpha}du = -\frac{\Gamma(2-\alpha)}{\alpha - 1}\cos(\alpha\pi/2)\text{sgn}(x)|x|^{\alpha-1}.$$

The latter two equalities imply the claim.

Recall that L_{T_x} is an exponential random variable with parameter $[h^*(x)]^{-1}$, then by (38), in the case when $(X, \mathbb{P})$ is an α-stable process with $\alpha \in (1, 2]$, L_{T_x} is an exponential random variable with parameter $1/[2K(\alpha)|x|^{\alpha-1}]$.

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